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Integrated Development of the Equations of Motion for Elastic Hypersonic Flight Vehicles

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An integrated, consistent analytical framework is developed for modeling the dynamics of elastic hypersonic flight vehicles. A Lagrangian approach is used to capture the dynamics of rigid-body motion, elastic deformation, fluid flow, rotating machinery, wind, and a spherical rotating Earth model and to account for their mutual interactions. The resulting equations of motion govern the rigid-body and elastic degrees of freedom (DOF). The elastic motion is represented in terms of modal displacement coordinates relative to the elastic mean axes system, and the rigid-body motion is represented in terms of the translational and rotational velocities of this axes system. A vector form of the force, moment, and elastic-deformation equations is developed from Lagrange's equation; a usable scalar form of these equations is also presented. The appropriate kinematic equations are developed and are presented in a usable form. The characteristics of the three-DOF point-mass dynamic model are also outlined, and the corresponding equations are presented. A preliminary study of the significance of selected terms in the equations of motion is conducted. Using generic data for a single-stage-to-orbit vehicle, it was found that the Coriolis force can reach values up to 6% of the vehicle weight and that the forces and moments attributable to fluid-flow terms can be significant.

I. Introduction

THE traditional development of the equations of motion for flight vehicles can be found in flight mechanics textbooks[1,2] where a Newtonian approach is used and the elastic degrees of freedom are excluded. On the other hand, structural dynamic models for flutter analysis frequently exclude some or all of the rigid-body degrees of freedom.[3] For vehicles with sufficiently stiff structures, the effects of elastic deformation can be included by assuming complete decoupling of the "rigid-body" and "elastic" modes. Noting that the separation between these modes is not large for vehicles characterized by significant aeroelastic effects, Waszak and Schmidt[4] used a Lagrangian approach to develop the equations of motion for elastic airplanes. Among the simplifications used in the development of Ref. 4 are a "flat, nonrotating Earth" model and the absence of fluid flow, wind, and rotating machinery.

For conventional aircraft, a traditional approach to dynamic modeling (e.g., considering airframe and engine as separate entities, using a rigid-body approximation, and then accounting for elastic effects by using quasistatic methods, etc.) can be used without introducing significant errors. Flight vehicles of the future, such as single-stage-to-orbit (SSTO) vehicles, are more likely to have complex interactions among their components, thereby motivating an integrated, multidisciplinary approach to their design and development.[5,6] Moreover, certain assumptions that are routinely made for conventional aircraft (e.g., nonrotating Earth, no fluid-flow effects, no aeroelastic effects) may not be valid for these classes of vehicles.

The objective of the present work is to develop an integrated, consistent analytical framework for modeling the dynamics of elastic hypersonic flight vehicles. The elastic deformation of the vehicle is modeled by using mode shape functions and in vacuo modal vibration frequencies obtained from structural dynamic analysis. Applying "first principles" to each particle of the vehicle, a Lagrangian approach is used to develop the force, moment, and elastic-deformation equations for the vehicle. The appropriate kinematic equations are also developed. A rigorous and unified approach yields a dynamic model that captures the effects of rigid-body motion, elastic deformation, fluid flow, rotating machinery, wind, and a spherical rotating Earth, and their mutual interactions. This dynamic model can be used for trajectory analysis and/or optimization and to analyze and design flight control systems. It may also be used to identify potential performance or stability-and-control problems in the early stages of conceptual design of the vehicle and to recommend changes in vehicle configuration to rectify such problems. Using generic data for a single-stage-to-orbit vehicle, a preliminary study is conducted to assess the significance of selected effects.

II. Lagrangian Approach

A. Vector Notation and Derivatives

The notation $A|^F$ indicates that the vector A is expressed in terms of its components along the axes of frame F, whereas the notation $\mathrm{d}A/\mathrm{d}t|_F$ indicates that the time rate of change of vector A is evaluated in frame F. The time derivative of vector A in frame 2 can be expressed in terms of the time derivative of vector A in frame 1, using the relationship

$$\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right|_2 = \left.\frac{\mathrm{d}A}{\mathrm{d}t}\right|_1 + \bar{\omega}_{1,2} \times A \tag{1}$$

where $\bar{\omega}_{1,2}$ is the angular velocity of frame 1 relative to frame 2.

B. Axes Systems and Transformations

The inertial frame I, Earth-fixed frame E, and vehicle-carrying frame V are defined in the standard fashion[1] and are shown in Fig. 1. It is noted that the Earth-center frame is assumed to be inertial; hence $\bar{\omega}_{E,I}$ is the angular velocity of the Earth about its axis of rotation. The body frame B is fixed to the body of the vehicle, with origin at the instantaneous center of mass (c.m.) of the vehicle.

The elastic mean axes system is chosen as the body axes system. Let r be the position vector of a mass element dm of the vehicle,

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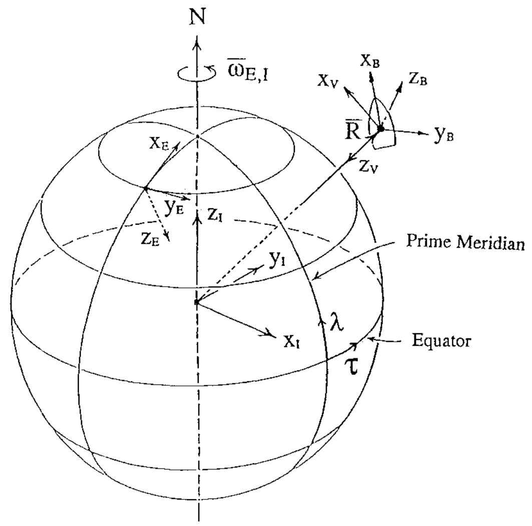

Fig. 1 Coordinate frames.

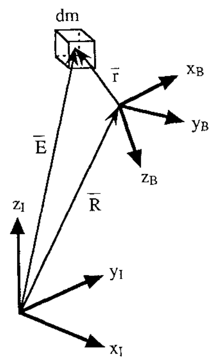

Fig. 2 Position of mass element.

measured from the origin of the body frame as shown in Fig. 2. For an elastic vehicle, this vector can be written as $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{e}$, where $\boldsymbol{r}_0$ is the position vector of mass element dm in the undeformed vehicle, and $\boldsymbol{e}$ is the change in position vector of the mass element dm due to elastic deformation of the vehicle. The mean axes are defined such that the relative linear and angular momenta, due to elastic deformation, are zero at every instant. Thus, the mean elastic body axes must satisfy the following condition[4]:

$$\int_m \left.\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t}\right|_B \mathrm{d}m = \bar{0} = \int_m \boldsymbol{r} \times \left.\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t}\right|_B \mathrm{d}m \tag{2}$$

The components of some vector $\boldsymbol{A}$ can be transformed from the vehicle-carrying frame to the body frame by using the relationship

$$\boldsymbol{A}|^B = [T]\boldsymbol{A}|^V \tag{3}$$

where $[T]$ is a coordinate transformation matrix. If the orientation of the body frame relative to the vehicle-carrying frame is described by the roll, pitch, and yaw angles (ϕ, θ, ψ) in the standard aircraft (3-2-1) Euler sequence, then the coordinate transformation matrix $[T]$ is given by[1]

$$[T] = \begin{bmatrix} c\theta\, c\psi & c\theta\, s\psi & -s\theta \\ \begin{matrix} s\phi\, s\theta\, c\psi \\ -c\phi\, s\psi \end{matrix} & \begin{matrix} s\phi\, s\theta\, s\psi \\ +c\phi\, c\psi \end{matrix} & s\phi\, c\theta \\ \begin{matrix} c\phi\, s\theta\, c\psi \\ +s\phi\, s\psi \end{matrix} & \begin{matrix} c\phi\, s\theta\, s\psi \\ -s\phi\, c\psi \end{matrix} & c\phi\, c\theta \end{bmatrix} \tag{4a}$$

Alternatively, the orientation of the body frame relative to the vehicle-carrying frame can be described by the quaternion components $\beta_i = \cos(\varepsilon_i)\sin(\delta/2)$ for $i = 1, 2, 3$, and $\beta_4 = \cos(\delta/2)$, where ε_i are the angles between the eigenaxis and the body axes, and δ is the principal rotation angle. The coordinate transformation matrix $[T]$ is given by[7]

$$[T] = \begin{bmatrix} \begin{pmatrix} \beta_1^2 - \beta_2^2 \\ -\beta_3^2 + \beta_4^2 \end{pmatrix} & 2\begin{pmatrix} \beta_1\beta_2 \\ +\beta_3\beta_4 \end{pmatrix} & 2\begin{pmatrix} \beta_1\beta_3 \\ -\beta_2\beta_4 \end{pmatrix} \\ 2\begin{pmatrix} \beta_1\beta_2 \\ -\beta_3\beta_4 \end{pmatrix} & \begin{pmatrix} -\beta_1^2 + \beta_2^2 \\ -\beta_3^2 + \beta_4^2 \end{pmatrix} & 2\begin{pmatrix} \beta_2\beta_3 \\ +\beta_1\beta_4 \end{pmatrix} \\ 2\begin{pmatrix} \beta_1\beta_3 \\ +\beta_2\beta_4 \end{pmatrix} & 2\begin{pmatrix} \beta_2\beta_3 \\ -\beta_1\beta_4 \end{pmatrix} & \begin{pmatrix} -\beta_1^2 - \beta_2^2 \\ +\beta_3^2 + \beta_4^2 \end{pmatrix} \end{bmatrix} \tag{4b}$$

C. Structural Model of Vehicle

It is assumed that the elastic deformation of the vehicle is sufficiently small and can be represented in terms of its normal undamped modes of free vibration. Hence, at a given mass element, the elastic deformation can be modeled as

$$\boldsymbol{e} = \sum_{i=1}^{\infty} \bar{\phi}_i \eta_i \tag{5}$$

where $\bar{\phi}_i$ are the mode shape functions, and η_i are generalized coordinates giving the magnitudes of the modal displacements. The values of the mode shape functions $(\bar{\phi}_i)$ at a mass element depend on the position of that mass element in the vehicle; the values of the generalized displacement coordinates (η_i) are strictly functions of time. For the purposes of structural modeling, the vehicle is divided into a large number of elements. The coordinates of each element in the undeformed vehicle are available from vehicle geometry. Structural dynamic analysis (e.g., finite element approach) provides the mode shape function components at each element of the vehicle, as well as the in vacuo modal vibration frequencies (ω_i), for a selected number of modes.

D. Lagrange's Equations of Motion

Lagrange's equations of motion[8] for a dynamic system can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\xi}_i}\right) - \left(\frac{\partial T}{\partial \xi_i}\right) + \left(\frac{\partial U}{\partial \xi_i}\right) = Q_i \tag{6}$$

where T is the kinetic energy, U is the potential energy, and ξ_i are generalized coordinates, Q_i are generalized forces given by

$$Q_i = \frac{\partial(\delta W)}{\partial(\delta \xi_i)} \tag{7}$$

where δW is the virtual work done on the system by all external forces/moments (excluding those already accounted for in the potential energy term) during a virtual displacement along all of the generalized coordinates.

The motion of the flight vehicle can be described in terms of the position $\bar{R}$ relative to the inertial frame (see Fig. 2), the orientation Γ relative to the inertial frame, and the modal displacement coordinates (η_i) representing elastic displacements relative to the undeformed shape. Hence the generalized coordinates may be written as

$$\bar{\xi}^T = \begin{bmatrix} \bar{\xi}_F^T & \bar{\xi}_M^T & \bar{\xi}_E^T \end{bmatrix} \tag{8}$$

where

$$\bar{\xi}_F = \boldsymbol{R}|^B = [R_x \quad R_y \quad R_z]^T \tag{9a}$$

$$\bar{\xi}_M = \boldsymbol{\Gamma}|^{Eu^*} = [\Phi \quad \Theta \quad \Psi]^T \tag{9b}$$

$$\bar{\xi}_E = [\eta_1 \quad \eta_2 \quad \eta_3 \quad \ldots]^T \tag{9c}$$

The distances (R_x, R_y, R_z) are components of the vector $\boldsymbol{R}$ along the body axes, and the angles (Φ, Θ, Ψ) are components of the vector

Γ along the Euler system of axes (Eu^*) for the (3-2-1) sequence of rotations from the inertial frame to the body frame. It is emphasized that these angles are different from the Euler angles (ϕ, θ, ψ), which are angles in the standard aircraft (3-2-1) sequence of rotations from the vehicle-carrying frame to the body frame, as defined earlier.

E. Kinetic Energy of the Vehicle

The kinetic energy of the vehicle (T) is equal to the sum of the kinetic energy of all mass elements of the vehicle and is given by

$$T = \frac{1}{2}\int_m \left.\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}\right|_I \cdot \left.\frac{\mathrm{d}\boldsymbol{E}}{\mathrm{d}t}\right|_I \mathrm{d}m \tag{10}$$

where $\boldsymbol{E}$ is the position vector of a mass element $\mathrm{d}m$ of the vehicle, as shown in Fig. 2. Noting that $\boldsymbol{E} = \boldsymbol{R} + \boldsymbol{r}$, and expanding, one obtains

$$T = \tfrac{1}{2}m(\boldsymbol{V}_I \cdot \boldsymbol{V}_I) + \bar{\boldsymbol{V}}_I \cdot \int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_I \mathrm{d}m + \frac{1}{2}\int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_I \cdot \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_I \mathrm{d}m \tag{11}$$

where $\boldsymbol{V}_I$ is the inertial velocity of the vehicle, and m is the instantaneous total mass of the vehicle.

Applying Eq. (1) to $\boldsymbol{r}$ and integrating over all mass elements of the vehicle yield

$$\int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_I \mathrm{d}m = \int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_B \mathrm{d}m + \boldsymbol{\omega}_{B,I} \times \int_m \boldsymbol{r}\,\mathrm{d}m \tag{12}$$

where $\boldsymbol{\omega}_{B,I}$ is the rotational velocity of the body frame relative to the inertial frame. The position vector $\boldsymbol{r}$ of a mass element $\mathrm{d}m$ is measured from the vehicle center of mass. Hence

$$\int_m \boldsymbol{r}\,\mathrm{d}m = \boldsymbol{0} \tag{13}$$

and therefore the second term on the right-hand side of Eq. (12) vanishes.

Taking the time derivative of Eq. (13), and noting that the mass of the vehicle (m) is changing due to fluids entering/leaving through openings in the vehicle, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_m \boldsymbol{r}\mathrm{d}m\right)\bigg|_B = \int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_B \mathrm{d}m + \sum_{\text{openings}} \dot{m}_{\text{fluid}}\,\boldsymbol{r}_{\text{open}} = \boldsymbol{0} \tag{14}$$

where $\dot{m}_{\text{fluid}}$ is the average mass flow rate of fluid entering (positive) or leaving (negative) the vehicle through an opening, and $\boldsymbol{r}_{\text{open}}$ is the average location of the opening relative to the origin of the body frame. The second term on the right-hand side of Eq. (11) can now be written as

$$\boldsymbol{V}_I \cdot \int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_I \mathrm{d}m = -\boldsymbol{V}_I \cdot \left(\sum_{\text{openings}} \dot{m}_{\text{fluid}}\,\boldsymbol{r}_{\text{open}}\right) \tag{15}$$

Applying Eq. (1) to $\boldsymbol{r}$, the third term on the right-hand side of Eq. (11) can be expanded to yield

$$\frac{1}{2}\int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_I \cdot \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_I \mathrm{d}m = \frac{1}{2}\int_m (\boldsymbol{\omega}_{B,I} \times \boldsymbol{r}) \cdot (\boldsymbol{\omega}_{B,I} \times \boldsymbol{r})\,\mathrm{d}m$$
$$+ \int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_B \cdot (\boldsymbol{\omega}_{B,I} \times \boldsymbol{r})\,\mathrm{d}m + \frac{1}{2}\int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_B \cdot \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_B \mathrm{d}m \tag{16}$$

Expanding terms in the integrand and integrating over all mass elements of the vehicle, the first term on the right-hand side of Eq. (16) can be written as

$$\frac{1}{2}\int_m (\boldsymbol{\omega}_{B,I} \times \boldsymbol{r}) \cdot (\boldsymbol{\omega}_{B,I} \times \boldsymbol{r})\,\mathrm{d}m = \tfrac{1}{2}\boldsymbol{\omega}_{B,I}^T[I]\boldsymbol{\omega}_{B,I} \tag{17}$$

where $[I]$ is the matrix of moments of inertia about the body (mean) axes of all mass elements of the deformed vehicle.

Using a result for scalar triple products, and noting that $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{e}$, the second term of Eq. (16) can be written as

$$\int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_B \cdot (\boldsymbol{\omega}_{B,I} \times \boldsymbol{r})\,\mathrm{d}m = \boldsymbol{\omega}_{B,I} \cdot \left(\int_m \boldsymbol{r} \times \left.\frac{\mathrm{d}\boldsymbol{r}_0}{\mathrm{d}t}\right|_B \mathrm{d}m \right.$$
$$\left. + \int_m \bar{r} \times \left.\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t}\right|_B \mathrm{d}m\right) \tag{18}$$

The second integral in the right-hand side of Eq. (18) is zero, from the definition of the mean axes system as given by Eq. (2); the first integral is zero for all mass elements of the vehicle that have no rigid-body motion relative to the body frame. Modeling all moving subsystems (e.g., rotating machinery) of the vehicle as rigid, balanced rotors, Eq. (18) can be written as

$$\int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_B \cdot (\boldsymbol{\omega}_{B,I} \times \boldsymbol{r})\,\mathrm{d}m = \boldsymbol{\omega}_{B,I} \cdot \left\{\sum_{\text{rotors}}([I_R]\boldsymbol{\omega}_{R,B})\right\} \tag{19}$$

where $[I_R]$ is the matrix of moments of inertia about the rotor axes, of all mass elements of a rotor, and $\boldsymbol{\omega}_{R,B}$ is the rotational velocity of the rotor frame relative to the body frame.

Using $\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{e}$, and noting the rigid-rotor assumption, the third term in Eq. (16) can be written as

$$\frac{1}{2}\int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_B \cdot \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_B \mathrm{d}m = \sum_{\text{rotors}}\left(\tfrac{1}{2}\boldsymbol{\omega}_{R,B}^T[I_R]\boldsymbol{\omega}_{R,B}\right)$$
$$+ \frac{1}{2}\int_m \left.\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t}\right|_B \cdot \left.\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t}\right|_B \mathrm{d}m \tag{20}$$

Using Eq. (5), the second term on the right-hand side of Eq. (20) can be written as

$$\frac{1}{2}\int_m \left.\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t}\right|_B \cdot \left.\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t}\right|_B \mathrm{d}m = \frac{1}{2}\sum_{i,j=1}^{\infty}\left\{\left[\int_m (\bar{\phi}_i \cdot \bar{\phi}_j)\,\mathrm{d}m\right](\dot{\eta}_i\dot{\eta}_j)\right\} \tag{21}$$

Since the modes of free vibration are normal (i.e., orthogonal), it follows that

$$\int_m (\bar{\phi}_i \cdot \bar{\phi}_j)\,\mathrm{d}m = \begin{cases} M_i & \text{for } j = i \\ 0 & \text{for } j \neq i \end{cases} \tag{22}$$

where M_i is the generalized modal mass associated with the ith elastic mode. Equation (20) can now be written as

$$\frac{1}{2}\int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_B \cdot \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_B \mathrm{d}m = \sum_{\text{rotors}}\left(\tfrac{1}{2}\boldsymbol{\omega}_{R,B}^T[I_R]\boldsymbol{\omega}_{R,B}\right) + \frac{1}{2}\sum_{i=1}^{\infty} M_i\dot{\eta}_i^2 \tag{23}$$

Substituting Eqs. (17), (19), and (23) into Eq. (16) results in

$$\frac{1}{2}\int_m \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_I \cdot \left.\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|_I \mathrm{d}m = \tfrac{1}{2}\boldsymbol{\omega}_{B,I}^T[I]\boldsymbol{\omega}_{B,I} + \frac{1}{2}\sum_{i=1}^{\infty} M_i\dot{\eta}_i^2$$
$$+ \boldsymbol{\omega}_{B,I}^T\left\{\sum_{\text{rotors}}[I_R]\boldsymbol{\omega}_{R,B}\right\} + \sum_{\text{rotors}}\left(\tfrac{1}{2}\boldsymbol{\omega}_{R,B}^T[I_R]\boldsymbol{\omega}_{R,B}\right) \tag{24}$$

Substituting Eqs. (15) and (24) into Eq. (11), one obtains the final expression for the kinetic energy of the vehicle:

$$T = \tfrac{1}{2}m\boldsymbol{V}_I^T\boldsymbol{V}_I + \tfrac{1}{2}\boldsymbol{\omega}_{B,I}^T[I]\boldsymbol{\omega}_{B,I} + \frac{1}{2}\sum_{i=1}^{\infty} M_i\dot{\eta}_i^2$$
$$- \boldsymbol{V}_I^T\left(\sum_{\text{openings}} \dot{m}_{\text{fluid}}\,\boldsymbol{r}_{\text{open}}\right) + \boldsymbol{\omega}_{B,I}^T\left(\sum_{\text{rotors}}[I_R]\boldsymbol{\omega}_{R,B}\right)$$
$$+ \sum_{\text{rotors}}\left(\tfrac{1}{2}\boldsymbol{\omega}_{R,B}^T[I_R]\boldsymbol{\omega}_{R,B}\right) \tag{25}$$

F. Potential Energy of the Vehicle

The potential energy of the vehicle consists of two parts and is given by

$$U = U_e + U_g \tag{26}$$

where U_e is the elastic strain energy of the vehicle, and U_g is the potential energy due to the gravitational field acting on the vehicle. The elastic strain energy may be written as[4]

$$U_e = -\frac{1}{2}\int_m \left[\left(\frac{d^2\boldsymbol{e}}{dt^2}\bigg|_B\right)\cdot \boldsymbol{e}\right]dm = \frac{1}{2}\sum_{i=1}^{\infty} M_i\omega_i^2\eta_i^2 \tag{27}$$

where ω_i is the in vacuo modal vibration frequency associated with the ith elastic mode.

The gravitational potential energy is given by

$$U_g = \int_m (\boldsymbol{R}+\boldsymbol{r})\cdot \boldsymbol{g}\, dm = m(\boldsymbol{g}^T\boldsymbol{R}) = -mg_0\frac{R^2_{\text{earth}}}{R} \tag{28}$$

where $\boldsymbol{g}$ is the absolute acceleration due to the Earth's gravitational field at the vehicle location, R_{earth} is the local radius of the Earth at the vehicle coordinates, and g_0 is the sea level value of the absolute acceleration due to the Earth's gravitational field at the vehicle coordinates. For a homogeneous spherical Earth model, the absolute sea level gravity value $g_0 = 9.80834$ (m/s^2) and the mean Earth radius value $R_{\text{earth}} = 6371$ (km) may be used.

G. Generalized Forces

The generalized forces in Lagrange's equation may be represented by

$$\boldsymbol{Q}^T = \left[\boldsymbol{Q}_F^T \quad \boldsymbol{Q}_M^T \quad \boldsymbol{Q}_E^T\right] \tag{29}$$

where

$$\boldsymbol{Q}_i = \left(\frac{\partial(\delta W)}{\partial(\delta\boldsymbol{\xi}_i)}\right)^T; \quad i = F, M, E \tag{30}$$

The virtual work (δW) done during a virtual displacement along the generalized coordinates ($\boldsymbol{R}, \boldsymbol{\Gamma}, \phi_i, \eta_i$) consists of the work done by the pressure distribution and fluid flow during the rigid-body translation and rotation of the vehicle and the work done by the pressure distribution during the elastic deformation of the vehicle. Hence,

$$\begin{aligned}\delta W = &\left\{\int_{A_{\text{vehicle}}}[(P-P_\infty)\hat{\boldsymbol{n}} + f\hat{\boldsymbol{t}}]\,dA + \sum_{\text{openings}}\left(\dot{m}_{\text{fluid}}\frac{d\boldsymbol{E}_{\text{fluid}}}{dt}\bigg|_I\right)\right\}\cdot\delta\boldsymbol{R}\\ &+\left\{\int_{A_{\text{vehicle}}}\boldsymbol{E}\times[(P-P_\infty)\hat{\boldsymbol{n}} + f\hat{\boldsymbol{t}}]\,dA + \sum_{\text{openings}}\boldsymbol{E}_{\text{open}}\times\left(\dot{m}_{\text{fluid}}\frac{d\boldsymbol{E}_{\text{fluid}}}{dt}\bigg|_I\right)\right\}\cdot\delta\boldsymbol{\Gamma}\\ &+\int_{A_{\text{vehicle}}}\left\{[(P-P_\infty)\hat{\boldsymbol{n}} + f\hat{\boldsymbol{t}}]\cdot\left[\sum_{i=1}^{\infty}\bar{\boldsymbol{\phi}}_i(\delta\eta_i)\right]\right\}dA\end{aligned} \tag{31}$$

where A_{vehicle} is the surface area of a control volume enclosing the vehicle, dA is an elemental area on the surface of the control volume, $\hat{\boldsymbol{n}}$ is an inward-pointing unit vector normal to dA, $\hat{\boldsymbol{t}}$ is a unit vector tangent to dA along the local flow direction, P is the local static pressure at element dA, P_∞ is the atmospheric (freestream) static pressure, f is the local tangential stress due to the flowfield at element dA, $\boldsymbol{E}$ is the position vector of element dA measured from the origin of the inertial frame, $\boldsymbol{E}_{\text{open}}$ is the average location of an opening measured from the inertial frame, and $\boldsymbol{E}_{\text{fluid}}$ is the average location of fluid elements at the threshold of an opening, measured from the inertial frame. It is noted that although $\boldsymbol{E}_{\text{fluid}} = \boldsymbol{E}_{\text{open}}$, their time derivatives are not equal in general. The vehicle control volume surface area can be written as

$$A_{\text{vehicle}} = A_w + \sum_{\text{openings}} A_{\text{open}} \tag{32}$$

where A_w is the wetted surface area of the vehicle, and A_{open} is the inlet/exit area, measured perpendicular to the internal-flow axis, of a rocket or air-breathing jet engine.

The generalized force $\boldsymbol{Q}_F$ is obtained by using Eqs. (30–32). Noting that $\boldsymbol{R}|^B = \boldsymbol{\xi}_F$, one obtains

$$\boldsymbol{Q}_F = \boldsymbol{F}_A + \boldsymbol{F}_T + \sum_{\text{openings}}\left[\dot{m}_{\text{fluid}}\left(\boldsymbol{V}_I + \frac{d\boldsymbol{r}_{\text{open}}}{dt}\bigg|_I\right)\right] \tag{33}$$

where $\boldsymbol{F}_A$ and $\boldsymbol{F}_T$ are the aerodynamic force and thrust force, respectively, given by

$$\boldsymbol{F}_A = \int_{A_w}[(P-P_\infty)\hat{\boldsymbol{n}} + f\hat{\boldsymbol{t}}]\,dA \tag{34}$$

$$\boldsymbol{F}_T = \sum_{\text{openings}}[A_{\text{open}}(P-P_\infty)\hat{\boldsymbol{n}}_{\text{open}} + \dot{m}_{\text{fluid}}\boldsymbol{V}_{f,o}] \tag{35}$$

In Eqs. (34) and (35), $\hat{\boldsymbol{n}}_{\text{open}}$ is an inward-pointing unit vector normal to A_{open}, and $\boldsymbol{V}_{f,o}$ is the velocity of the fluid relative to the opening. It is noted that the force resulting from flow turning at an engine inlet/exit has been included in the $\boldsymbol{F}_T$ term and that the mass flow rate $\dot{m}_{\text{fluid}}$ may be modeled as a function of the local angle of attack, the deformed shape of the inlet/exit, and the relative fluid velocity $\boldsymbol{V}_{f,o}$.

An expression for the generalized force $\boldsymbol{Q}_M$ is obtained by using Eqs. (30–32). Noting that $\boldsymbol{\Gamma}|^B = [G]\boldsymbol{\Gamma}|^{Eu^*} = [G]\boldsymbol{\xi}_M$, one obtains

$$\begin{aligned}\boldsymbol{Q}_M = [G]^T \times \Bigg\{&\boldsymbol{M}_A + \boldsymbol{M}_T + \sum_{\text{openings}}\left(\dot{m}_{\text{fluid}}\boldsymbol{r}_{\text{open}}\times\left[\boldsymbol{V}_I + \frac{d\boldsymbol{r}_{\text{open}}}{dt}\bigg|_I\right]\right)\\ &+\boldsymbol{R}\times\left[\boldsymbol{F}_A + \boldsymbol{F}_T + \sum_{\text{openings}}\left(\dot{m}_{\text{fluid}}\left[\boldsymbol{V}_I + \frac{d\boldsymbol{r}_{\text{open}}}{dt}\bigg|_I\right]\right)\right]\Bigg\}\end{aligned} \tag{36a}$$

where $[G]$ is a coordinate transformation matrix given by

$$[G] = \begin{bmatrix}1 & 0 & -\sin\Theta\\ 0 & \cos\Phi & \sin\Phi\cos\Theta\\ 0 & -\sin\Phi & \cos\Phi\cos\Theta\end{bmatrix} \tag{36b}$$

The quantities $\boldsymbol{M}_A$ and $\boldsymbol{M}_T$ are the aerodynamic moment and thrust moment, respectively, about the vehicle c.m. and are given by

$$\boldsymbol{M}_A = \int_{A_w}[(P-P_\infty)(\boldsymbol{r}\times\hat{\boldsymbol{n}}) + f(\boldsymbol{r}\times\hat{\boldsymbol{t}})]\,dA \tag{37}$$

$$\boldsymbol{M}_T = \sum_{\text{openings}}[\boldsymbol{r}_{\text{open}}\times(A_{\text{open}}(P-P_\infty)\hat{\boldsymbol{n}}_{\text{open}} + \dot{m}_{\text{fluid}}\boldsymbol{V}_{f,o})] \tag{38}$$

It is noted that the moment resulting from flow turning at an engine inlet/exit has been included in the $\boldsymbol{M}_T$ term.

The generalized aeroelastic forces Q_{E_i} are obtained by using Eqs. (30) and (31). Noting that $\xi_{E_i} = \phi_i\eta_i$, one obtains

$$Q_{E_i} = \int_{A_w}[(P-P_\infty)\hat{\boldsymbol{n}} + f\hat{\boldsymbol{t}}]\cdot\bar{\boldsymbol{\phi}}_i dA; \quad i = 1, 2, 3, \ldots \tag{39}$$

Reference 9, for example, presents a simplified aeropropulsive and aeroelastic model. It includes the development of expressions for $\boldsymbol{F}_A, \boldsymbol{F}_T, \boldsymbol{M}_A, \boldsymbol{M}_T$, and P, for a generic hypersonic vehicle.

III. Force and Moment Equations

A. Force Equations

From Lagrange's equation, the force equation of motion (EOM) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\xi}_F}\right) - \left(\frac{\partial T}{\partial \xi_F}\right) + \left(\frac{\partial U}{\partial \xi_F}\right) = \boldsymbol{Q}_F \tag{40}$$

The force EOM is constructed by substituting Eqs. (25–28) and (33) into Eq. (40). After some algebraic manipulation, one obtains

$$m\left.\frac{\mathrm{d}\boldsymbol{V}_I}{\mathrm{d}t}\right|_I = m\boldsymbol{g} + \boldsymbol{F}_A + \boldsymbol{F}_T + \sum_{\text{openings}}\left[2\dot{m}_{\text{fluid}}\left(\left.\frac{\mathrm{d}\boldsymbol{r}_{\text{open}}}{\mathrm{d}t}\right|_B + \boldsymbol{\omega}_{B,I}\times\boldsymbol{r}_{\text{open}}\right) + \ddot{m}_{\text{fluid}}\boldsymbol{r}_{\text{open}}\right] \tag{41}$$

It is noted that the general form of the force equation given by Eq. (41) is similar to that obtained by using a Newtonian approach and the solidification principle.[2]

Using Eq. (1), the inertial acceleration of the vehicle can be expressed as

$$\left.\frac{\mathrm{d}\boldsymbol{V}_I}{\mathrm{d}t}\right|_I = \left.\frac{\mathrm{d}\boldsymbol{V}_E}{\mathrm{d}t}\right|_E + 2\boldsymbol{\omega}_{E,I}\times\boldsymbol{V}_E + \boldsymbol{\omega}_{E,I}\times(\boldsymbol{\omega}_{E,I}\times\boldsymbol{R}) \tag{42}$$

where $\boldsymbol{V}_E$ is the vehicle velocity relative to the Earth-fixed frame and is equal to the vector sum of the vehicle velocity relative to the atmosphere ($\boldsymbol{V}_A$) and the velocity of the atmosphere relative to the Earth ($\boldsymbol{V}_{\text{wind}}$). Hence $\boldsymbol{V}_E = (\boldsymbol{V}_A + \boldsymbol{V}_{\text{wind}})$, and Eq. (42) can be written as

$$\left.\frac{\mathrm{d}\boldsymbol{V}_I}{\mathrm{d}t}\right|_I = \left.\frac{\mathrm{d}\boldsymbol{V}_A}{\mathrm{d}t}\right|_B + \boldsymbol{\omega}_{B,I}\times\boldsymbol{V}_A + \boldsymbol{\omega}_{E,I}\times\boldsymbol{V}_A + \boldsymbol{\omega}_{E,I}\times(\boldsymbol{\omega}_{E,I}\times\boldsymbol{R}) + \left.\frac{\mathrm{d}\boldsymbol{V}_{\text{wind}}}{\mathrm{d}t}\right|_E + 2\boldsymbol{\omega}_{E,I}\times\boldsymbol{V}_{\text{wind}} \tag{43}$$

Substituting Eq. (43) into Eq. (41), the vector form of the force EOM can be written in an expanded form, as presented in Fig. 3.

To write the force EOM in scalar form, the appropriate vector quantities must be expressed in terms of scalar components along the body axes. The vehicle airspeed $\boldsymbol{V}_A = [u\ v\ w]^T$, the vehicle absolute angular velocity $\boldsymbol{\omega}_{B,I} = [p\ q\ r]^T$, the inlet locations $\boldsymbol{r}_{\text{in}} = [x_{\text{in}}\ y_{\text{in}}\ z_{\text{in}}]^T$, the exit locations $\boldsymbol{r}_{\text{out}} = [x_{\text{out}}\ y_{\text{out}}\ z_{\text{out}}]^T$, the aerodynamic force $\boldsymbol{F}_A = [X_A\ Y_A\ Z_A]^T$, and the thrust force $\boldsymbol{F}_T = [X_T\ Y_T\ Z_T]^T$ are expressed directly in terms of components along the body axes. The vehicle position vector $\boldsymbol{R} = [0\ 0\ -R]^T$, the Earth's gravity vector $\boldsymbol{g} = [0\ 0\ g_0(R_{\text{earth}}/R)^2]^T$, the Earth's absolute angular velocity $\boldsymbol{\omega}_{E,I} = \omega_{\text{earth}}[\cos\lambda\ 0\ -\sin\lambda]^T$, and the Earth-relative wind velocity $\boldsymbol{V}_{\text{wind}} = [W_N\ W_E\ -W_U]^T$ are first expressed in terms of components along the vehicle-carrying axes, as indicated earlier, and then transformed into components

$$m\left\{\left.\frac{\mathrm{d}\bar{V}_A}{\mathrm{d}t}\right|_B + \bar{\omega}_{B,I}\times\bar{V}_A\right\}$$

Rigid-body terms

$$+\ m\left\{\bar{\omega}_{E,I}\times\bar{V}_A + \bar{\omega}_{E,I}\times(\bar{\omega}_{E,I}\times\bar{R})\right\}$$

Rotating-Earth terms
Interaction: Rigid-body, Altitude

$$+\ m\left\{\left.\frac{\mathrm{d}\bar{V}_{\text{wind}}}{\mathrm{d}t}\right|_E + 2\,\bar{\omega}_{E,I}\times\bar{V}_{\text{wind}}\right\}$$

Wind terms
Interaction: Rotating-Earth, Longitude, Latitude, Altitude

$$+\ \sum_{\text{engines}}\left\{\begin{aligned}&2\,\dot{m}_{\text{air}}\left[\bar{\omega}_{B,I}\times(\bar{r}_{\text{out}}-\bar{r}_{\text{in}}) + \left(\left.\frac{\mathrm{d}\bar{r}_{\text{out}}}{\mathrm{d}t}\right|_B - \left.\frac{\mathrm{d}\bar{r}_{\text{in}}}{\mathrm{d}t}\right|_B\right)\right] + \ddot{m}_{\text{air}}(\bar{r}_{\text{out}}-\bar{r}_{\text{in}})\\ &+ 2\,\dot{m}_{\text{fuel}}\left[\bar{\omega}_{B,I}\times\bar{r}_{\text{out}} + \left.\frac{\mathrm{d}\bar{r}_{\text{out}}}{\mathrm{d}t}\right|_B\right] + \ddot{m}_{\text{fuel}}\,\bar{r}_{\text{out}}\end{aligned}\right\}$$

Fluid-flow terms
Interaction:
Rigid-body,
Elastic-deformation

$$=\ m\,\bar{g} + \bar{F}_A + \bar{F}_T$$

Gravitational, Aerodynamic and Propulsive force terms
Interaction: Rigid-body, Fluid-flow, Elastic-deformation, Altitude

$$[I]\left.\frac{\mathrm{d}\bar{\omega}_{B,I}}{\mathrm{d}t}\right|_B + \bar{\omega}_{B,I}\times[I]\,\bar{\omega}_{B,I} + [\dot{I}]\,\bar{\omega}_{B,I}$$

Rigid-body terms
Interaction: Fluid-flow, Elastic-deformation

$$+\ \sum_{\text{rotors}}\left(\bar{\omega}_{B,I}\times[I_R]\,\bar{\omega}_{R,B} + [I_R]\left.\frac{\mathrm{d}\bar{\omega}_{R,B}}{\mathrm{d}t}\right|_B\right)$$

Rotor terms
Interaction: Rigid-body

$$+\ \sum_{\text{engines}}\left\{\begin{aligned}&\dot{m}_{\text{air}}\left[\bar{r}_{\text{out}}\times(\bar{\omega}_{B,I}\times\bar{r}_{\text{out}}) - \bar{r}_{\text{in}}\times(\bar{\omega}_{B,I}\times\bar{r}_{\text{in}}) + \bar{r}_{\text{out}}\times\left.\frac{\mathrm{d}\bar{r}_{\text{out}}}{\mathrm{d}t}\right|_B - \bar{r}_{\text{in}}\times\left.\frac{\mathrm{d}\bar{r}_{\text{in}}}{\mathrm{d}t}\right|_B\right]\\ &+ \dot{m}_{\text{fuel}}\left[\bar{r}_{\text{out}}\times(\bar{\omega}_{B,I}\times\bar{r}_{\text{out}}) + \bar{r}_{\text{out}}\times\left.\frac{\mathrm{d}\bar{r}_{\text{out}}}{\mathrm{d}t}\right|_B\right]\end{aligned}\right\}$$

Fluid-flow terms
Interaction:
Rigid-body,
Elastic
-deformation

$$=\ \bar{M}_A + \bar{M}_T$$

Aerodynamic and Propulsive moment terms
Interaction: Rigid-body, Fluid-flow, Elastic-deformation, Altitude

$$M_i\,(\ddot{\eta}_i + 2\,\zeta_i\,\omega_i\,\dot{\eta}_i + \omega_i^2\eta_i)$$

Elastic-deformation terms

$$=\ \int_{A_w}\left[(P-P_\infty)\,\hat{n} + f\,\hat{t}\right]^T\bar{\phi}_i\,\mathrm{d}A$$

Aeroelastic force term
Interaction: Rigid-body, Altitude

Fig. 3 Force, moment and elastic-deformation equations.

Table 1 Force equations in body axes system

$$m(\dot{u}+qw-rv)+m\left\{\begin{array}{l}\omega_{\text{earth}}[w(T_{21}\cos\lambda-T_{23}\sin\lambda)-v(T_{31}\cos\lambda-T_{33}\sin\lambda)]\\ +R\omega^2_{\text{earth}}(T_{11}\cos\lambda\,\sin\lambda+T_{13}\cos^2\lambda)\end{array}\right\}$$

$$+m\left\{\begin{array}{l}(\dot{W}_N T_{11}+\dot{W}_E T_{12}-\dot{W}_U T_{13})\\ +2\omega_{\text{earth}}[W_N(-T_{12}\sin\lambda)+W_E(T_{11}\sin\lambda+T_{13}\cos\lambda)+W_U(T_{12}\cos\lambda)]\end{array}\right\}$$

$$+\sum_{\text{engines}}\left\{\begin{array}{l}2\dot{m}_{\text{air}}[q(z_{\text{out}}-z_{\text{in}})-r(y_{\text{out}}-y_{\text{in}})+(\dot{x}_{\text{out}}-\dot{x}_{\text{in}})]+\ddot{m}_{\text{air}}(x_{\text{out}}-x_{\text{in}})\\ +2\dot{m}_{\text{fuel}}[qz_{\text{out}}-ry_{\text{out}}+\dot{x}_{\text{out}}]+\ddot{m}_{\text{fuel}}x_{\text{out}}\end{array}\right\}$$

$$=mg_0\left(\frac{R_{\text{earth}}}{R}\right)^2 T_{13}+X_A+X_T$$

$$m(\dot{v}+ru-pw)+m\left\{\begin{array}{l}\omega_{\text{earth}}[u(T_{31}\cos\lambda-T_{33}\sin\lambda)-w(T_{11}\cos\lambda-T_{13}\sin\lambda)]\\ +R\omega^2_{\text{earth}}(T_{21}\cos\lambda\,\sin\lambda+T_{23}\cos^2\lambda)\end{array}\right\}$$

$$+m\left\{\begin{array}{l}(\dot{W}_N T_{21}+\dot{W}_E T_{22}-\dot{W}_U T_{23})\\ +2w_{\text{earth}}[W_N(-T_{22}\sin\lambda)+W_E(T_{21}\sin\lambda+T_{23}\cos\lambda)+W_U(T_{22}\cos\lambda)]\end{array}\right\}$$

$$+\sum_{\text{engines}}\left\{\begin{array}{l}2\dot{m}_{\text{air}}[r(x_{\text{out}}-x_{\text{in}})-p(z_{\text{out}}-z_{\text{in}})+(\dot{y}_{\text{out}}-\dot{y}_{\text{in}})]+\ddot{m}_{\text{air}}(y_{\text{out}}-y_{\text{in}})\\ +2\dot{m}_{\text{fuel}}[rx_{\text{out}}-pz_{\text{out}}+\dot{y}_{\text{out}}]+\ddot{m}_{\text{fuel}}y_{\text{out}}\end{array}\right\}$$

$$=mg_0\left(\frac{R_{\text{earth}}}{R}\right)^2 T_{23}+Y_A+Y_T$$

$$m(\dot{w}+pv-qu)+m\left\{\begin{array}{l}\omega_{\text{earth}}[v(T_{11}\cos\lambda-T_{13}\sin\lambda)-u(T_{21}\cos\lambda-T_{23}\sin\lambda)]\\ +R\omega^2_{\text{earth}}(T_{31}\cos\lambda\,\sin\lambda+T_{33}\cos^2\lambda)\end{array}\right\}$$

$$+m\left\{\begin{array}{l}(\dot{W}_N T_{31}+\dot{W}_E T_{32}-\dot{W}_U T_{33})\\ +2\omega_{\text{earth}}[W_N(-T_{32}\sin\lambda)+W_E(T_{31}\sin\lambda+T_{33}\cos\lambda)+W_U(T_{32}\cos\lambda)]\end{array}\right\}$$

$$+\sum_{\text{engines}}\left\{\begin{array}{l}2\dot{m}_{\text{air}}[p(y_{\text{out}}-y_{\text{in}})-q(x_{\text{out}}-x_{\text{in}})+(\dot{z}_{\text{out}}-\dot{z}_{\text{in}})]+\ddot{m}_{\text{air}}(z_{\text{out}}-z_{\text{in}})\\ +2\dot{m}_{\text{fuel}}[py_{\text{out}}-qx_{\text{out}}+\dot{z}_{\text{out}}]+\ddot{m}_{\text{fuel}}z_{\text{out}}\end{array}\right\}$$

$$=mg_0\left(\frac{R_{\text{earth}}}{R}\right)^2 T_{33}+Z_A+Z_T$$

along the body axes by using the coordinate transformation matrix $[T]$ given by Eq. (4). After expressing all vector quantities in the force equation in terms of their scalar components, three scalar force equations along the body axes are obtained; these equations are presented in Table 1. It is noted that the mass flow rates of air and fuel in the force equations are positive quantities since the appropriate signs have already been absorbed into the force EOM.

B. Moment Equations

From Lagrange's equation, the moment EOM is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\dot{\xi}_M}\right)-\left(\frac{\partial T}{\partial\xi_M}\right)+\left(\frac{\partial U}{\partial\xi_M}\right)=\boldsymbol{Q}_M \tag{44}$$

The absolute angular momentum (about vehicle c.m.) of all mass elements of the vehicle is given by

$$\boldsymbol{H}=[I]\boldsymbol{\omega}_{B,I}+\sum_{\text{rotors}}[I_R]\boldsymbol{\omega}_{R,B} \tag{45}$$

The moment EOM is constructed by substituting Eqs. (25–28) and (36) into Eq. (44). After considerable algebraic manipulation, one obtains

$$\left.\frac{\mathrm{d}\boldsymbol{H}}{\mathrm{d}t}\right|_I=\boldsymbol{M}_A+\boldsymbol{M}_T+\sum_{\text{openings}}\left[\dot{m}_{\text{fluid}}\boldsymbol{r}_{\text{open}}\times\left(\left.\frac{\mathrm{d}\boldsymbol{r}_{\text{open}}}{\mathrm{d}t}\right|_B+\boldsymbol{\omega}_{B,I}\times\boldsymbol{r}_{\text{open}}\right)\right] \tag{46}$$

It is noted that the general form of the moment equation given by Eq. (46) is similar to that obtained by using a Newtonian approach and the solidification principle.[2]

Substituting Eq. (45) into Eq. (46), and applying Eq. (1) to $\boldsymbol{\omega}_{B,I}$ and $\boldsymbol{\omega}_{R,B}$, the vector form of the force EOM can be written in an expanded form, as presented in Fig. 3.

It is recalled that $[I]$ is the inertia matrix for the entire vehicle and can be expressed as

$$[I]=\begin{bmatrix}I_x & -I_{xy} & -I_{xz}\\ -I_{xy} & I_y & -I_{yz}\\ -I_{xz} & -I_{yz} & I_z\end{bmatrix} \tag{47}$$

Changes in the inertia matrix can be caused by changes in mass distribution due to net fluid flow through the vehicle and by elastic deformation. The latter may be neglected if the elastic deformations are assumed to be sufficiently small.

It is assumed that the rotors have an axisymmetric mass distribution and rotate about their axes of symmetry. The mass moment of inertia about the axis of rotation is I_{rotor}, and the rotor angular velocity relative to the body frame is ω_{rotor}. The orientation of a rotor axis relative to the body axes system is given by $(\varepsilon_{\text{rotor}},\sigma_{\text{rotor}})$ where $\varepsilon_{\text{rotor}}$ is the angle between the rotor axis and the body x-z plane, and σ_{rotor} is the angle between the body x axis and the projection of the rotor axis on the body x-z plane. Hence the rotor angular momentum can be expressed in terms of components along body axes as $[I_R]\boldsymbol{\omega}_{R,B}=I_{\text{rotor}}\omega_{\text{rotor}}[(c\varepsilon_{\text{rotor}}\,c\sigma_{\text{rotor}})\ (s\varepsilon_{\text{rotor}})\ (c\varepsilon_{\text{rotor}}\,s\sigma_{\text{rotor}})]^T$. The aerodynamic moment and thrust moment vectors can be expressed in terms of components along the body axes as $\boldsymbol{M}_A=[L_A\ M_A\ N_A]^T$ and $\boldsymbol{M}_T=[L_T\ M_T\ N_T]^T$, respectively. The remaining vectors in the moment equation are expressed in terms of components along the body axes, as described in Sec. III. A. After expressing all vector quantities in the moment equation in terms of their scalar components, three scalar moment equations along the body axes are obtained; these equations are presented in Table 2. It is noted that the mass flow rates of air and fuel in the moment equations are positive quantities since the appropriate signs have already been absorbed into the moment EOM.

Table 2 Moment equations in body axes system

$$
\begin{aligned}
& I_x\dot p + (I_z - I_y)qr - I_{xz}(\dot r + pq) - I_{xy}(\dot q - pr) + I_{yz}(r^2 - q^2) + \dot I_x p - \dot I_{xy} q - \dot I_{xz} r \\
& + \sum_{\text{rotors}} \{I_{\text{rotor}}[\omega_{\text{rotor}}(q\cos\varepsilon_{\text{rotor}}\sin\sigma_{\text{rotor}} - r\sin\varepsilon_{\text{rotor}}) + \dot\omega_{\text{rotor}}\cos\varepsilon_{\text{rotor}}\cos\sigma_{\text{rotor}}]\} \\
& + \sum_{\text{engines}} \left\{ \begin{array}{l} \dot m_{\text{air}}\left[\begin{array}{l} \left\{\left(y_{\text{out}}^2 - y_{\text{in}}^2\right) + \left(z_{\text{out}}^2 - z_{\text{in}}^2\right)\right\}p - (x_{\text{out}}y_{\text{out}} - x_{\text{in}}y_{\text{in}})q \\ -(x_{\text{out}}z_{\text{out}} - x_{\text{in}}z_{\text{in}})r + (y_{\text{out}}\dot z_{\text{out}} - y_{\text{in}}\dot z_{\text{in}}) - (z_{\text{out}}\dot y_{\text{out}} - z_{\text{in}}\dot y_{\text{in}}) \end{array}\right] \\ + \dot m_{\text{fuel}}\left[\left(y_{\text{out}}^2 + z_{\text{out}}^2\right)p - (x_{\text{out}}y_{\text{out}})q - (x_{\text{out}}z_{\text{out}})r + y_{\text{out}}\dot z_{\text{out}} - z_{\text{out}}\dot y_{\text{out}}\right] \end{array} \right\} \\
& = L_A + L_T
\end{aligned}
$$

$$
\begin{aligned}
& I_y\dot q + (I_x - I_z)pr + I_{xz}(p^2 - r^2) - I_{xy}(\dot p + qr) - I_{yz}(\dot r - pq) - \dot I_{xy} p + \dot I_y q - \dot I_{yz} r \\
& + \sum_{\text{rotors}} \{I_{\text{rotor}}[\omega_{\text{rotor}}(r\cos\varepsilon_{\text{rotor}}\cos\sigma_{\text{rotor}} - p\cos\varepsilon_{\text{rotor}}\sin\sigma_{\text{rotor}}) + \dot\omega_{\text{rotor}}\sin\varepsilon_{\text{rotor}}]\} \\
& + \sum_{\text{engines}} \left\{ \begin{array}{l} \dot m_{\text{air}}\left[\begin{array}{l} -(x_{\text{out}}y_{\text{out}} - x_{\text{in}}y_{\text{in}})p + \left\{\left(x_{\text{out}}^2 - x_{\text{in}}^2\right) + \left(z_{\text{out}}^2 - z_{\text{in}}^2\right)\right\}q \\ -(y_{\text{out}}z_{\text{out}} - y_{\text{in}}z_{\text{in}})r + (z_{\text{out}}\dot x_{\text{out}} - z_{\text{in}}\dot x_{\text{in}}) - (x_{\text{out}}\dot z_{\text{out}} - x_{\text{in}}\dot z_{\text{in}}) \end{array}\right] \\ + \dot m_{\text{fuel}}\left[-(x_{\text{out}}y_{\text{out}})p + \left(x_{\text{out}}^2 + z_{\text{out}}^2\right)q - (y_{\text{out}}z_{\text{out}})r + z_{\text{out}}\dot x_{\text{out}} - x_{\text{out}}\dot z_{\text{out}}\right] \end{array} \right\} \\
& = M_A + M_T
\end{aligned}
$$

$$
\begin{aligned}
& I_z\dot r + (I_y - I_x)pq - I_{xz}(\dot p - qr) + I_{xy}(q^2 - p^2) - I_{yz}(\dot q + pr) - \dot I_{xz} p - \dot I_{yz} q + \dot I_z r \\
& + \sum_{\text{rotors}} \{I_{\text{rotor}}[\omega_{\text{rotor}}(p\sin\varepsilon_{\text{rotor}} - q\cos\varepsilon_{\text{rotor}}\cos\sigma_{\text{rotor}}) + \dot\omega_{\text{rotor}}\cos\varepsilon_{\text{rotor}}\sin\sigma_{\text{rotor}}]\} \\
& + \sum_{\text{engines}} \left\{ \begin{array}{l} \dot m_{\text{air}}\left[\begin{array}{l} -(x_{\text{out}}z_{\text{out}} - x_{\text{in}}z_{\text{in}})p + \left\{\left(x_{\text{out}}^2 - x_{\text{in}}^2\right) + \left(y_{\text{out}}^2 - y_{\text{in}}^2\right)\right\}r \\ -(y_{\text{out}}z_{\text{out}} - y_{\text{in}}z_{\text{in}})q + (x_{\text{out}}\dot y_{\text{out}} - x_{\text{in}}\dot y_{\text{in}}) - (y_{\text{out}}\dot x_{\text{out}} - y_{\text{in}}\dot x_{\text{in}}) \end{array}\right] \\ + \dot m_{\text{fuel}}\left[-(x_{\text{out}}z_{\text{out}})p - (y_{\text{out}}z_{\text{out}})q + \left(x_{\text{out}}^2 + y_{\text{out}}^2\right)r + x_{\text{out}}\dot y_{\text{out}} - y_{\text{out}}\dot x_{\text{out}}\right] \end{array} \right\} \\
& = N_A + N_T
\end{aligned}
$$

C. Elastic-Deformation Equations

From Lagrange's equation, the elastic-deformation EOM is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot\xi_E}\right)^T - \left(\frac{\partial T}{\partial \xi_E}\right)^T + \left(\frac{\partial U}{\partial \xi_E}\right)^T = Q_E \tag{48}$$

The elastic-deformation EOM is constructed by substituting Eqs. (25–28) into Eq. (48) and is given by

$$\left(\ddot\eta_i + \omega_i^2\eta_i = \frac{Q_{E_i}}{M_i}\right); \quad i = 1, 2, 3, \ldots \tag{49}$$

Although the structural damping has been neglected in this analysis, the effects of damping can be included a posteriori by adding a damping term characterized by the damping ratio ζ_i. This value is (Ref. 1, p. 173) "ordinarily less than 0.1, and usually must be found by an experimental measurement on the actual structure." Thus the modified elastic-deformation EOM may be written as

$$\left(\ddot\eta_i + 2\zeta_i\omega_i\dot\eta_i + \omega_i^2\eta_i = \frac{Q_{E_i}}{M_i}\right); \quad i = 1, 2, 3, \ldots \tag{50}$$

Substituting Eq. (39) into Eq. (50), the elastic-deformation EOM can be written in an expanded form, as presented in Fig. 3.

IV. Kinematic Equations

The kinematic equations consist of two sets of equations: the trajectory equations that describe the position of the vehicle relative to the Earth-fixed frame and the Euler angle equations (or alternatively, the quaternion equations) that describe the orientation of the vehicle relative to the vehicle-carrying frame. The development of these equations is outlined below, and a final usable form of the kinematic equations is presented in Table 3.

A. Trajectory Equations

The trajectory equations are relationships that transform components of the vector V_E along body axes into components along vehicle-carrying axes, where $V_E = (V_A + V_{\text{wind}})$ is the Earth-relative velocity of the vehicle. Hence

$$V_E|^V = [T]^{-1}V_E|^B = [T]^T V_A|^B + V_{\text{wind}}|^V \tag{51}$$

Expressing the vectors in Eq. (51) in terms of the appropriate scalar components, one obtains the trajectory equation

$$\begin{bmatrix} R\dot\lambda \\ (R\cos\lambda)\dot\tau \\ -\dot R \end{bmatrix} = [T]^T \begin{bmatrix} u \\ v \\ v \end{bmatrix} + \begin{bmatrix} W_N \\ W_E \\ -W_U \end{bmatrix} \tag{52}$$

where λ, τ, and R are the vehicle's longitude, latitude, and distance from Earth center, respectively (see Fig. 2). The entries of the coordinate transformation matrix $[T]$ are given by Eq. (4a) or by Eq. (4b), as appropriate.

B. Euler Angle and Quaternion Equations

The Euler angle equations are relationships that transform components of the vector $\omega_{B,V}$ along body axes into components along the Euler system of axes (Eu) in the standard aircraft (3-2-1) sequence of rotations from the vehicle-carrying frame to the body frame; $\omega_{B,V}$ is the angular velocity of the body frame relative to the vehicle-carrying frame. Thus

$$\omega_{B,V}|^{Eu} = [L]\omega_{B,V}|^B \tag{53}$$

where $[L]$ is a coordinate transformation matrix from body axes to the Euler system of axes. Expressing the vectors in Eq. (53) in terms of the appropriate scalar components,[1] one obtains the Euler angle equations:

$$\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{bmatrix} \begin{bmatrix} p_v \\ q_v \\ r_v \end{bmatrix} \tag{54a}$$

where

$$\begin{bmatrix} p_v \\ q_v \\ r_v \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} - [T] \begin{bmatrix} (\dot\tau + \omega_{\text{earth}})\cos\lambda \\ -\dot\lambda \\ -(\dot\tau + \omega_{\text{earth}})\sin\lambda \end{bmatrix} \tag{54b}$$

In Eq. (54b), $\dot\lambda$ and $\dot\tau$ are obtained from Eq. (52), and the entries of the coordinate transformation matrix $[T]$ are taken from Eq. (4a).

If quaternion components are used to represent the orientation of the body frame relative to the vehicle-carrying frame,[7] one obtains

Table 3 Kinematic equations

Trajectory equations

$$\dot{\lambda} = \frac{1}{R}[(T_{11}u + T_{21}v + T_{31}w) + W_N]$$
$$\dot{\tau} = \frac{\sec\lambda}{R}[(T_{12}u + T_{22}v + T_{32}w) + W_E]$$
$$\dot{R} = [-(T_{13}u + T_{23}v + T_{33}w) + W_U]$$

Euler angle equations

$$\dot{\phi} = p_v + q_v \sin\phi \tan\theta + r_v \cos\phi \tan\theta$$
$$\dot{\theta} = q_v \cos\phi - r_v \sin\phi$$
$$\dot{\psi} = q_v \sin\phi \sec\theta + r_v \cos\phi \sec\theta$$

Quaternion equations

$$\dot{\beta}_1 = \tfrac{1}{2}(p_v\beta_4 - q_v\beta_3 + r_v\beta_2)$$
$$\dot{\beta}_2 = \tfrac{1}{2}(p_v\beta_3 + q_v\beta_4 - r_v\beta_1)$$
$$\dot{\beta}_3 = \tfrac{1}{2}(-p_v\beta_2 + q_v\beta_1 + r_v\beta_4)$$
$$\dot{\beta}_4 = \tfrac{1}{2}(-p_v\beta_1 - q_v\beta_2 - r_v\beta_3)$$

where

$$p_v = p + \frac{1}{R}[u(T_{12}T_{13}\tan\lambda) + v(T_{13}T_{22}\tan\lambda + T_{12}T_{21} - T_{11}T_{22}) + w(T_{13}T_{32}\tan\lambda + T_{12}T_{31} - T_{11}T_{32})] + \omega_{\text{earth}}(T_{13}\sin\lambda - T_{11}\cos\lambda) + \frac{1}{R}[W_N T_{12} + W_E(T_{13}\tan\lambda - T_{11})]$$

$$q_v = q + \frac{1}{R}[u(T_{12}T_{23}\tan\lambda + T_{11}T_{22} - T_{12}T_{21}) + v(T_{22}T_{23}\tan\lambda) + w(T_{23}T_{32}\tan\lambda + T_{22}T_{31} - T_{21}T_{32})] + \omega_{\text{earth}}(T_{23}\sin\lambda - T_{21}\cos\lambda) + \frac{1}{R}[W_N T_{22} + W_E(T_{23}\tan\lambda - T_{21})]$$

$$r_v = r + \frac{1}{R}[u(T_{12}T_{33}\tan\lambda + T_{11}T_{32} - T_{12}T_{31}) + v(T_{22}T_{33}\tan\lambda + T_{21}T_{32} - T_{22}T_{31}) + w(T_{32}T_{33}\tan\lambda)] + \omega_{\text{earth}}(T_{33}\sin\lambda - T_{31}\cos\lambda) + \frac{1}{R}[W_N T_{32} + W_E(T_{33}\tan\lambda - T_{31})]$$

the quaternion equations

$$\begin{bmatrix}\dot{\beta}_1\\ \dot{\beta}_2\\ \dot{\beta}_3\\ \dot{\beta}_4\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0 & r_v & -q_v & p_v\\ -r_v & 0 & p_v & q_v\\ q_v & -p_v & 0 & r_v\\ -p_v & -q_v & -r_v & 0\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\end{bmatrix} \tag{55}$$

where (p_v, q_v, r_v) are given by Eq. (54b). The entries of the coordinate transformation matrix $[T]$ in Eq. (54b) are taken from Eq. (4b).

V. Point-Mass Model

The EOM can also be used to derive other dynamic models, such as the linearized EOM and the three-DOF point-mass model. The point-mass model is typically used in trajectory studies such as performance analysis and optimization, where the translational motion of the vehicle is of primary interest. This model consists of the (nonlinear) force and trajectory equations written in a special form. The moment equations do not appear in this three-DOF model. The decoupling of the force and moment equations is based on the assumption that control surface deflections do not significantly alter the aerodynamic forces appearing in the force equations. The moment equations may be employed a posteriori to predict the control surface deflections required to maintain the vehicle along the path described by the force equations.[2] A final usable form of this model is presented in Table 4. In these equations, T is the total thrust force of all of the engines, D is the drag force, Q is the side force (along the negative y axis), α_T is the average thrust angle of attack of all of the engines, and β_T is the average thrust sideslip angle of all of the engines. It is noted that the wind axes angles (μ, γ, χ) are used in lieu of the standard Euler angles (ϕ, θ, ψ), where μ is the bank angle, γ is the flight-path angle, and χ is the heading angle. The fluid-flow terms, which depend primarily on the vehicle angular velocity $\boldsymbol{\omega}_{B,I}$, do not appear in Table 4 because the point-mass model does not include rotational dynamics.

VI. Discussion of Vehicle Equations of Motion

A survey of the vehicle EOM shows that there are a total of $(12 + n)$ equations, where n is the number of structural modes included in the model. The breakdown of the EOM is as follows: three force equations, three moment equations, n elastic-deformation equations, three trajectory equations, and three Euler angle equations (or, alternatively, four quaternion equations).

From an inspection of the EOM presented in Tables 1–3, the coupling between the force, moment, trajectory, and Euler angle (or quaternion) equations is evident. The aerodynamic forces and moments on the right-hand sides of the force and moment equations have a strong dependence on the elastic deformation; the inlet/exit locations are also affected by elastic deformation. Additionally, the aeroelastic forces on the right-hand sides of the elastic-deformation equations have a strong dependence on the vehicle velocities (translational and rotational). Hence the elastic-deformation equations are coupled with the force and moment equations. There is also an additional inertial coupling of the moment equations with the elastic-deformation equations through the moment-of-inertia derivative terms, but this coupling is very weak for small elastic deformations.

Figure 3 highlights the various effects (rigid body, rotating Earth, wind, fluid flow, elastic deformation, rotating machinery, vehicle position) that affect the dynamics of the vehicle; the interactions between these effects are also noted. It is not feasible to reach a universal conclusion as to which effects and/or interactions are negligible, since the EOM developed in this work are applicable to a wide variety of flight vehicles. However, the information contained in Fig. 3 permits the user to obtain a reasonable estimate of several interacting terms. It should be noted that the dynamic model developed in this work does not explicitly include fuel slosh effects (which could be significant for SSTO vehicles); however, they may be included by expanding the definition of the mean body axes.[1]

The EOM can be used for various applications such as dynamic simulation and control system design. Depending on the application of interest (e.g., trajectory optimization, design of pitch autopilot), the Coriolis, transport, rotor, fluid-flow, and elastic-deformation effects may have varying degrees of importance. For a vehicle flying at Mach 25 at 200,000 ft altitude, the maximum magnitude of the Coriolis force $m\boldsymbol{\omega}_{E,I} \times \boldsymbol{V}_A$ is approximately 6% of the vehicle weight, and the maximum magnitude of the transport force $m\boldsymbol{\omega}_{E,I} \times (\boldsymbol{\omega}_{E,I} \times \boldsymbol{R})$ is approximately 0.35% of the vehicle weight. Using generic data for a single-stage-to-orbit vehicle ($W = 250,000$ lb, $\dot{m}_{\text{fluid}} = 500$ slug/s, $|\boldsymbol{r}_{\text{open}}| = 25$ ft) flying at Mach 8 at 80,000 ft altitude with an absolute angular velocity of 3 deg/s, the force associated with the fluid-flow term is of the order of 0.5% of the vehicle weight. It is estimated that the moment associated with the fluid-flow term is of the order of 1% of the maximum pitching moment that the elevator is capable of providing. These estimates of fluid-flow terms are based on a preliminary analysis[9] of the engine module only; in particular, the magnitude of $\boldsymbol{r}_{\text{open}}$ is an estimate of the distance between the vehicle center of mass and the actual nozzle of

Table 4 Point-mass model

$$\dot{\lambda} = \frac{1}{R}(V \cos\gamma \cos\chi + W_N)$$

$$\dot{\tau} = \frac{1}{R\cos\lambda}(V \cos\gamma \sin\chi + W_E)$$

$$\dot{R} = V \sin\gamma + W_U$$

$$\dot{V} = \left[\frac{T}{m}\cos\alpha_T \cos\beta_T - \frac{D}{m} - g_0\left(\frac{R_{\text{earth}}}{R}\right)^2 \sin\gamma\right] + \omega_{\text{earth}}^2 R(s\gamma c^2\lambda - c\gamma c\chi s\lambda c\lambda)$$
$$+ \frac{V}{R}[-W_N s\gamma c\gamma c\chi] + (-\dot{W}_N c\gamma c\chi - \dot{W}_E c\gamma s\chi - \dot{W}_U s\gamma)$$
$$+ 2\omega_{\text{earth}}[W_N c\gamma s\chi s\lambda + W_E(s\gamma c\lambda - c\gamma c\chi s\lambda) - W_U c\gamma s\chi c\lambda]$$

$$\dot{\gamma} = \frac{1}{V}\left\{\left[\frac{L}{m}\cos\mu + \frac{T}{m}(\sin\alpha_T \cos\beta_T \cos\mu + \sin\beta_T \sin\mu) + \frac{Q}{m}\sin\mu - g_0\left(\frac{R_{\text{earth}}}{R}\right)^2 \cos\gamma\right]\right.$$
$$+ \frac{V^2}{R}c\gamma + 2\omega_{\text{earth}} V s\chi c\lambda + \omega_{\text{earth}}^2 R(s\gamma c\chi s\lambda c\lambda + c\gamma c^2\lambda)$$
$$+ \frac{V}{R}\left[W_N s^2\gamma c\chi + W_E s\chi\right] + (\dot{W}_N s\gamma c\chi + \dot{W}_E s\gamma s\chi - \dot{W}_U c\gamma)$$
$$\left. + 2\omega_{\text{earth}}[-W_N s\gamma s\chi s\lambda + W_E(c\gamma c\lambda + s\gamma c\chi s\lambda) + W_U s\gamma s\chi c\lambda]\right\}$$

$$\dot{\chi} = \frac{1}{V\cos\gamma}\left\{\left[\frac{L}{m}\sin\mu + \frac{T}{m}(\sin\alpha_T \cos\beta_T \sin\mu - \sin\beta_T \cos\mu) - \frac{Q}{m}\cos\mu\right]\right.$$
$$+ \frac{V^2}{R}c^2\gamma s\chi t\lambda + 2\omega_{\text{earth}} V(c\gamma s\lambda - s\gamma c\chi c\lambda) + \omega_{\text{earth}}^2 R s\chi s\lambda c\lambda$$
$$+ \frac{V}{R}[W_N s\gamma s\chi + W_E(c\gamma t\lambda - s\gamma c\chi)]$$
$$\left. + (\dot{W}_N s\chi - \dot{W}_E c\chi) + 2\omega_{\text{earth}}[W_N c\chi s\lambda + W_E s\chi s\lambda - W_U c\chi c\lambda]\right\}$$

the engine module. It should be noted that the preceding estimates of the force and moment due to fluid-flow terms are for exit flow only.

VII. Conclusions

An integrated, consistent analytical framework has been developed for modeling the dynamics of elastic hypersonic flight vehicles. The dynamic model includes the effects of rigid-body motion, elastic deformation, fluid flow, rotating machinery, wind, and spherical rotating Earth, and their interactions with each other. Beginning with the application of "first principles" to each particle of the vehicle, a Lagrangian approach was used to develop the force, moment, and elastic-deformation equations in vector form. Scalar forms of these equations were then obtained. The appropriate kinematic equations were also developed and presented. The characteristics of the three-DOF point-mass model were outlined, and the corresponding equations were presented. A preliminary study of the significance of selected terms in the equations of motion was conducted. Using generic data for a single-stage-to-orbit vehicle, it was found that the Coriolis force can reach values up to 6% of the vehicle weight and that the forces and moments attributable to fluid-flow terms can be significant.

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